Comments on 'Gain Scheduling Dynamic Linear Controllers for a Nonlinear Plant'

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Abstract Lawrence & Rugh (1995), Kaminer *et al.* (1995) propose that the realisation for a gain-scheduled controller should be chosen to satisfy a local linear equivalence condition. However, this provides an inadequate basis selecting an appropriate realisation. Many realisations satisfy the local linear equivalence condition yet are not equivalent and can exhibit quite different dynamic behaviour. Furthermore, the condition imposes restrictions on the controller states and inputs which are not *a priori* necessary.

1. *Introduction*

A gain-scheduled controller is constructed by interpolating between the members of a family of linear controllers. However, the dynamic behaviour of the resulting controller, being nonlinear, can be strongly dependent on the realisation adopted (see, for example, Leith & Leithead 1996). Lawrence & Rugh (1995) propose that the realisation for a gain-scheduled controller should be chosen to satisfy a local linear equivalence condition; that is, the linearisation, at an equilibrium operating point, of the gain-scheduled controller should correspond to the associated member of the family of linear controllers. A similar requirement is also considered, in a more restricted context, by Kaminer *et al.* (1995).

2. *Local linear equivalence at equilibrium operating points*

Consider a SISO gain-scheduled controller described by the nonlinear differential equation

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, r, \rho); \quad y = H(\mathbf{x}, r, \rho) \tag{1}$$

where $\mathbf{F}(\bullet,\bullet,\bullet)$, $H(\bullet,\bullet,\bullet)$ are differentiable, r denotes the input to the system, y the output and ρ is a continuous scalar function of ($\mathbf{x}$, r) corresponding to the scheduling variable. The set of equilibrium operating points consists of those points, ($\mathbf{x}_o$, r_o), for which $\mathbf{F}(\mathbf{x}_o, r_o, \rho_o)$ is zero, where ρ_o denotes $\rho(\mathbf{x}_o, r_o)$. Let Φ: $\Re^n \times \Re$ denote the space $\{(\mathbf{x}, r)\}$. The set of equilibrium operating points forms a locus in Φ parameterised by ρ and the response of the controller to the general time-varying input, r(t), corresponds to a trajectory in Φ. Satisfying local linear equivalence at the equilibrium operating points, as

suggested by Lawrence & Rugh (1995), ensures that the gain-scheduled controller has similar stability properties to the appropriate linear controller near a specific equilibrium operating point only if ($\mathbf{x}$, r) (and so also $\rho(\mathbf{x}$,r)) remains within a sufficiently small neighbourhood of that operating point. Outwith each neighbourhood the gain-scheduled controller can exhibit very different characteristics from the local linearisation. The situation is illustrated by Fig. 1a for a SISO first-order controller: the neighbourhoods, depicted about specific equilibrium operating points, notionally indicate the respective regions within which linearisation is valid.

The choice of realisation is, however, not unique and different realisations are not equivalent. Although realisations satisfying the local linear equivalence condition of Lawrence & Rugh (1995) are not distinguishable at the equilibrium operating points, the size of the neighbourhoods of the equilibrium operating points, within which they are dynamically similar to the associated linear systems, can vary substantially. Indeed, the neighbourhoods could, in general, be vanishingly small. Nevertheless, Lawrence & Rugh (1995) do not distinguish between different controller realisations satisfying the local linear equivalence condition at equilibrium operating points.

For example, both of the realisations depicted in Fig. 2 satisfy the local linear equivalence condition at equilibrium operating points. The dynamic behaviour of the second order nonlinear element in Fig. 2b is described by the differential equation

$$\ddot{y} + a(u)\dot{y} + b(u)y = x \tag{2}$$

Locally to an equilibrium operating point at which the nominal value of the scheduling variable, u, is u_o,

$$u=u_o+\delta u;\ x=0+\delta x;\ y=0+\delta y;\ \dot{y} = 0+\delta\dot{y};\ \ddot{y} = 0+\delta\ddot{y} \tag{3}$$

and equation (2) has the linearisation

$$\delta\ddot{y} + a(u_o)\delta\dot{y} + b(u_o)\delta y \approx \delta x \tag{4}$$

From (4), it is clear that the local linear equivalence condition is satisfied at the equilibrium operating points. The error, ε, in approximating (2), locally to the equilibrium operating point, by (4) is

$$\varepsilon = \Big(a(u) - a(u_o)\Big)\delta\dot{y} + \Big(b(u) - b(u_o)\Big)\delta y \tag{5}$$

When $a(\bullet)$ and $b(\bullet)$ are continuous and non-zero for all possible values of u, it follows that ε can be made arbitrarily small for δu sufficiently small (and δy and its derivative finite); that is, the linearisation accurately describes the dynamic behaviour of (2) in an arbitrarily large neighbourhood in state-space about the equilibrium point provided δu is sufficiently small. Now, consider the realisation of Fig. 2a. The differential equation (2) is superseded by

$$\ddot{y} + a(u)\dot{y} + b(u)y + \frac{da}{du}\dot{u}\,y = x \tag{6}$$

Locally to an equilibrium operating point, the differential equation (6) can again be approximated linearly by (4) and the local linear equivalence condition is again satisfied at the equilibrium operating points. The error, ξ, in approximating (5), locally to the equilibrium point, by (4) is

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$$\xi = \left(a(u) - a(u_o)\right)\delta\dot{y} + \left(b(u) - b(u_o)\right)\delta y + \frac{da}{du}(u)\delta\dot{u}\,\delta y \qquad (7)$$

In general, $\delta\dot{u}$ can be arbitrarily large independently of the magnitude of δu and so, unless δy is confined to an infinitesimally small neighbourhood about the origin, ξ may be large even if δu is small. (Although it is unlikely that $\delta\dot{u}$ is unduly large for ideally deterministic systems, to which Lawrence & Rugh (1995) is restricted, this possibility is not so unlikely for stochastic systems). In practice, therefore, the neighbourhoods of the equilibrium operating points, within which (2) and (6) are dynamically similar to (4), may be quite different. Indeed, (6) may exhibit substantially different dynamic behaviour from (2) and from that indicated by its linearisation local to an equilibrium operating point, even when δu is small and the states of the system are rather close to their equilibrium values. By inspection, the realisation of Fig. 2b might seem to be preferable since the error, ξ, does not involve $\delta\dot{u}$. Surprisingly, however, a controller with a realisation similar to that of Fig. 2a can substantially out-perform one with a realisation similar to that of Fig. 2b (Leith & Leithead 1996); that is, the 'obvious' choice of realisation need not be the most appropriate.

Whilst it clearly remains attractive to require local linear equivalence at the equilibrium point, it is evident from the foregoing discussion that, by itself, this requirement provides an inadequate basis for the choice of realisation for gain-scheduled controllers. In order to address this deficiency, Leith & Leithead (1996) propose an extended local linear equivalence requirement. Whilst this extended requirement cannot be discussed in detail here, some salient points are highlighted.

Enclosing each equilibrium operating point, there is a neighbourhood within which the linearisation, obtained by perturbing the system about the equilibrium operating point, of the dynamics is valid. The equilibrium operating points and the associated neighbourhoods are parameterised by the scheduling variable, ρ[‡]. The local linear equivalence condition of Lawrence & Rugh 1995) neglects the dependence of these neighbourhoods on the choice of realisation. However, the neighbourhoods may be relatively large for some choices of realisation and vanishingly small for others. It is, therefore, attractive to extend the local linear equivalence condition to exclude realisations for which the neighbourhoods are unnecessarily small. In particular, for realisations satisfying the extended local linear equivalence condition of Leith & Leithead (1996), the neighbourhoods are sufficiently large that the *union* of them encompasses the whole solution space. In addition, any point in this space, for which ρ has the value ρ_o, is in the neighbourhood associated with the equilibrium operating point for that value of ρ_o. Hence, at any non-equilibrium operating point the dynamics can be linearised by associating them with the linear dynamics at the equilibrium operating point which has the same value of ρ. (Note, the linearisation is not obtained by perturbing the system about the non-equilibrium operating point and neglecting the inhomogeneous term, although that would result in a similar description for those systems satisfying the extended local linear equivalence condition). This linearisation is valid in any neighbourhood of the non-equilibrium operating point which is contained within the neighbourhood of the corresponding equilibrium operating point. Since linear analysis is then applicable to any trajectory for which ρ is constant, the extended condition corresponds to the natural requirement that, when ρ equals ρ_o, the dynamic behaviour is identical to the member, specified by ρ_o, of the family of linear controllers. Local linear equivalence in this extended sense ensures that the family of linear systems indicate the local dynamic behaviour at every point in Φ rather than only in a small region close to the locus of equilibrium operating points. Since the behaviour of the gain-scheduled controller is specified at every point in Φ, the extended local linear equivalence condition leads to an, essentially, unique choice of controller realisation. In Fig. 1b, the shaded region notionally indicates the proposed extended neighbourhood of linear equivalence about a specific surface on which ρ equals ρ_o. Each surface of constant ρ, and its corresponding neighbourhood, extends indefinitely in Φ, and the collection of surfaces and neighbourhoods covers the whole space, Φ.

The local linear equivalence condition of Lawrence & Rugh (1995) is, in general, confined to a only a small neighbourhood of the equilibrium operating points. Hence, it requires that every trajectory in Φ remains within a sufficiently small neighbourhood of the locus of equilibrium points. This requirement does not seem to be *a priori* necessary yet it limits the analysis to small (perhaps vanishingly small) neighbourhoods of the equilibrium

[‡] To avoid misunderstanding, it is emphasised that the dynamic behaviour of a gain-scheduled system depends, in general, on the behaviour of *every* member of the associated linearisation family. Unless the trajectories of the system are confined solely to a single extended neighbourhood, it is insufficient to consider a single member of the linearisation family in isolation. For example, consider the system $\dot{x}=(\rho^2-1)x+x^2-x^3+r$ which, when ρ equals x, may be reformulated as $\dot{x}=-x+x^2+r$ (note, for this particular choice of ρ the system satisfies the extended local linear equivalence condition but this is not the case for arbitrary ρ). The linearisation of these dynamics about the equilibrium point at which x equals 0 is stable, but at nearby equilibrium points (at which $x>½$) the linearisation is unstable. Hence, unless the input and initial conditions are constrained such that the solutions to the system are confined to the stable neighbourhoods the system is clearly unstable. (For example, with initial state $x(0)=0$ and constant input $r=½$, the solution of the linearisation about the equilibrium point at which x equals zero is $x(t)=½(1-e^{-t})\approx½t-¼t^2$ for small t. In comparison, the solution of the nonlinear system is $x(t)=½(1+\tan(t/2-\pi/4))\approx½t-¼t^2$ for small t. Clearly, the solutions agree initially but eventually diverge once they leave the neighbourhood within which the linearisation is valid).

operating points and as a consequence, from (1), *every* system state must be, in some sense, slowly-varying. In general, this implicitly imposes a constraint on the rate of variation of ρ which is likely to be quite restrictive. The extended local linear equivalence condition enables this restriction to be avoided. Realisations which satisfy the extended local linear equivalence condition are derived by Leith & Leithead (1996) for a wide class of SISO systems and it is observed that a controller realisation satisfying the extended condition can substantially out-perform realisations which only satisfy local linear equivalence about the equilibrium operating points and not the extended criterion (Leith & Leithead 1996). The extended local linear equivalence condition can, therefore, provide guidance as to what is an appropriate realisation for a gain-scheduled controller.

3. *Conclusions*

The approach of Lawrence & Rugh (1995) provides an inadequate basis for the selection of an appropriate controller realisation. Many different controller realisations satisfy the local linear equivalence condition, yet they are not equivalent and can, even when close to an equilibrium operating point, exhibit quite different dynamic behaviour. Furthermore, the condition can lead, in general, to the imposition of restrictive constraints on the controller states and inputs which are not *a priori* necessary.

References

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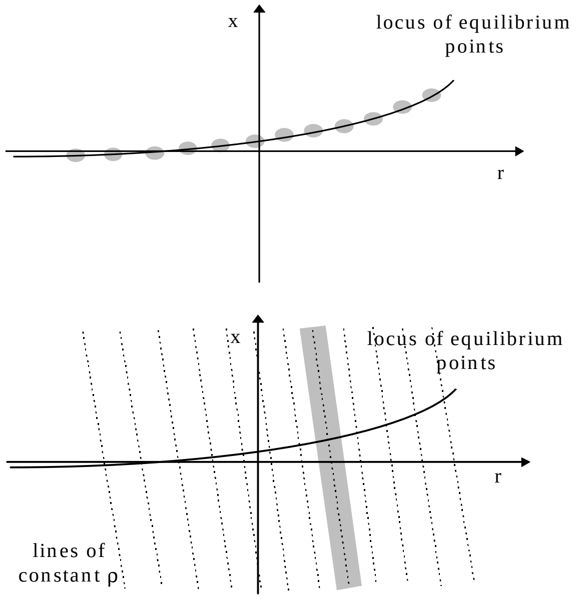

Figure 1. *(a)* Illustration of local linear equivalence neighbourhoods about equilibrium operating points. *(b)* Illustration of extended local linear equivalence neighbourhoods.

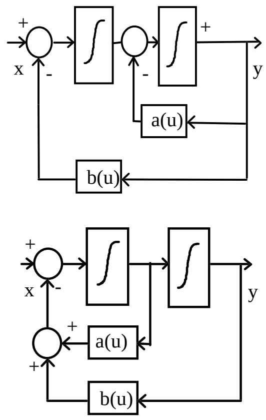

Figure 2. Example of different realisations satisfying local linear equivalence at equilibrium operating points.

RESPONSE TO LAWRENCE AND RUGH

In their response to Leith & Leithead (1998), Lawrence & Rugh make a number of comments regarding the results in Leith & Leithead (1996). We consider their remarks in turn.

1. Gain-scheduling is sometimes applied in conditions when it is not *a priori* warranted; that is, when the response of the system is not confined locally to a specific equilibrium operating point and/or the system is not slowly varying, e.g. wind turbine regulation as in Leith & Leithead (1996). It is observed that, in these circumstances, the choice of realisation of the controller contributes significantly to the performance of the nonlinear gain-scheduled controller and the choice of realisation must be considered with some care. The work of Lawrence & Rugh is discussed since it also considers the choice of realisation of the controller. However, in the context considered in Leith & Leithead, the work of Lawrence & Rugh does not assist in distinguishing between those controllers, of the many which meet their requirement, that perform well and those that perform less well. The inadequacy is in this lack of guidance rather than the lack of uniqueness.

2. Some restrictions on the performance attained by a gain-scheduled controller are inherent to the design task itself; for example, the restrictions induced by the degree of variation of the local plant dynamics along the locus of equilibrium operating points. Other restrictions arise from the choice of controller realisation; for example, the nonlinear nature of the controller away from the locus of equilibrium operating points. In particular, suppose that the neighbourhoods of each equilibrium operating point, within which the linearised dynamics are considered to be an adequate representation of the nonlinear gain-scheduled controller, are small then the solution trajectories must remain close to the equilibrium operating points. By choosing a realisation which satisfies an extended local linear equivalence condition, the latter unnecessary restriction is avoided but of course the inherent restrictions are not. Lawrence & Rugh comment that the analysis in Leith & Leithead (1996) contains no mathematical proof that the realisations satisfying the extended linear equivalence condition realise their purpose. However, the chosen realisations self-evidently achieve the stated aim since they ensure unbounded linearisation neighbourhoods the union of which covers the entire space; no proof is required.

Requiring the controllers to be of the form proposed in Leith & Leithead (1996), certainly eliminates the worst realisations, specifically, those with excessively small linearisation neighbourhoods, and should generally provide some guidance in identifying the better ones. It is not claimed that controllers not satisfying the condition can never out-perform controllers satisfying it. Indeed, controller designs which exploit full knowledge of the plant dynamics can lead to improved performance in comparison to the situation considered in Leith & Leithead (1996) where equilibrium information only is utilised.

3. Lawrence & Rugh quote short sections from Leith & Leithead (1996) out of context. Such “sound bites” are potentially misleading, particularly with regard to the quotation from the Conclusions section of Leith & Leithead (1996). The linearisation associated with a non-equilibrium operating point is explicitly addressed in both Leith & Leithead (1996) and Leith & Leithead (1998) and there is nothing “vague” about this concept. Specifically, the scheduling variable can be evaluated at any operating point. For the class of plants considered it has the same value at some equilibrium operating point. Furthermore, the neighbourhood, within which the linearised dynamics at the corresponding equilibrium operating point are valid, encloses the initially considered operating point. Hence, locally to the operating point the dynamics of the nonlinear gain-scheduled controller are described by the linear dynamics corresponding to the value of the scheduling variable. Since the neighbourhoods are unbounded, all operating points along any trajectory, even those far from the locus of equilibrium operating points, exhibit this dynamical equivalence. In other words, this dynamical equivalence does not require slow variation or confinement to the equilibrium operating points. A fuller quotation of the text in Leith & Leithead (1996) would have made this intended meaning clear.

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